THEORY OF AUTOMATA & FORMAL LANGUAGES

Operations on Strings

Transforming and examining sequences of symbols

Concatenate → Repeat → Reverse → Identify

Concatenation: joining strings

Concatenation places one string immediately after another, preserving symbol order.

Definition and example

Let u = ab and v = baa.

Write u first: ab

Append v: baa

uv = ab · baa = abbaa

|uv| = |u| + |v| = 2 + 3 = 5

Order and empty string

vu = baa · ab = baaab

Here uv ≠ vu.

Concatenation is associative:

(uv)w = u(vw)

uε = εu = u

Concatenation joins strings; it does not sort, merge or remove repeated symbols.

String powers: repeated concatenation

For a string w, the power w^n means concatenating n copies of w, where $n \geq 0$.

Build powers step by step

Let $w = ab$.

$w^0 = \varepsilon$

$w^1 = ab$

$w^2 = ab \cdot ab = abab$

$w^3 = ab \cdot ab \cdot ab = ababab$

Properties and example

$|w^n| = n \times |w|$

$|(ab)^3| = 3 \times 2 = 6$

$w^m w^n = w^{m+n}$

$(ab)^2(ab)^3 = (ab)^5$

$\varepsilon^n = \varepsilon$ for every $n \geq 0$

A string power repeats the whole block: $(ab)^3 \neq aaabbb$.

Reversal: reading a string backwards

The reverse w^R lists all symbols of w in the opposite order.

Manual example

Let w = aabb.

Original: a a b b

Read from right to left.

Reverse: b b a a

w^R = bbaa

Properties and examples

$|w^R| = |w|$

$(w^R)^R = w$

$(uv)^R = v^R u^R$

$(ab \cdot bb)^R = bb \cdot ba = bbba$

$\varepsilon^R = \varepsilon$

A palindrome equals its reverse; for example, $aba^R = aba$.

Prefixes: beginning of a string

A string x is a prefix of w if w = xy for some string y (possibly empty).

List all prefixes

Let w = abba.

Take zero symbols: ε

Take one symbol: a

Take two: ab; three: abb

Take four: abba

Membership and proper prefixes

Is ab a prefix? Yes: ab · ba.

Is ba a prefix? No.

Prefixes: {ε, a, ab, abb, abba}

Proper prefixes exclude abba.

A length-n string has n + 1 prefixes.

A prefix must start at the first position; it cannot skip symbols.

Suffixes: ending of a string

A string y is a suffix of w if w = xy for some string x (possibly empty).

List all suffixes

Let w = abba.

Take zero symbols: ε

Take last symbol: a

Take last two: ba; three: bba

Take all four: abba

Membership and proper suffixes

Is ba a suffix? Yes: ab · ba.

Is ab a suffix? No.

Suffixes: {ε, a, ba, bba, abba}

Proper suffixes exclude abba.

A length-n string has n + 1 suffixes.

A suffix must end at the last position; keep the original reading order.

Substrings: contiguous blocks

A string x is a substring of w if w = uxv for some strings u and v.

Find substrings of abba

Length 0: ε

Length 1: a, b

Length 2: ab, bb, ba

Length 3: abb, bba

Length 4: abba

Check the condition

Is bb a substring? Yes.

abba = a · bb · a

Is aa a substring? No.

aa is a subsequence: skip both b’s.

A substring cannot skip symbols.

Distinct substrings may repeat at different positions; count each distinct string once.

Compare the three kinds of blocks

Use the same example w = abba to see the difference between the three concepts.

Classify each candidate

ab: prefix and substring

ba: suffix and substring

bb: substring only

abba: all three

ε: all three

Important relationships

Every prefix is a substring.

Every suffix is a substring.

Not every substring is a prefix.

Not every substring is a suffix.

aa: none of the three.

Prefixes and suffixes are anchored at an end; substrings can occur anywhere.

Practice and explained answers

Use $\Sigma = \{a, b\}$. Let $u = ab$, $v = ba$ and $w = aab$.

Try these questions

1. Find uv and vu.
2. Find u^3 and $|u^3|$.
3. Find w^R.
4. List every prefix of w.
5. Is ab a suffix and substring of w?

Answers and reasoning

1. $uv = abba$; $vu = baab$.
2. $ababab$; length $= 6$.
3. baa: reverse the order.
4. ε, a, aa, aab
5. Yes to both: $w = a \cdot ab$.

Next: build formal languages by collecting strings that satisfy a common condition.