Pumping Lemma Solved Examples

Proof workshop

Choose a witness • constrain every split • pump • derive a contradiction

Purpose

Develop reusable strategies for proving that languages are not regular.

Prerequisite

Lecture 31: Pumping Lemma for Regular Languages.

Learning outcomes

Choose

Select witness strings that force the pumped part into a useful region.

Prove

Handle every legal decomposition, not merely one convenient split.

Evaluate

Use closure properties when a direct pumping proof is awkward.

Recall the proof framework

Assumption

Suppose L is regular. Let $p \geq 1$ be its pumping length.

Witness and split

Choose $w \in L$ with $|w| \geq p$. For any $w = xyz$ with $|xy| \leq p$ and $|y| \geq 1$, constrain y.

Contradiction

Find $i \geq 0$ such that $xy^iz \notin L$. This contradicts the lemma; L is not regular.

Quantifiers: who chooses what?

Opponent

Supplies p, the pumping length promised by regularity.

You

Choose w after knowing p. The opponent may choose any legal split x, y, z.

You again

Choose i, possibly depending on the split. Every legal split must fail for some i.

Example 1: unequal growth rates

Language

$L = \{0^n 1^{2n} : n \geq 0\}$.

Witness

Assume regularity with pumping length p. Choose $w = 0^p 1^{2p}$.

Legal split

The first p symbols are zeros, so $y = 0^k$ for some $1 \leq k \leq p$.

Example 1: complete the proof

Pump down

Take $i = 0$. Then $xz = 0^{p-k}1^{2p}$.

Check the language

There are $p - k$ zeros but $2p$ ones. Membership requires $2(p - k)$ ones.

Conclusion

Since $k \geq 1$, $2p \neq 2(p - k)$. Thus $xz \notin L$, contradicting the lemma.

Example 2: three equal blocks

Language

$L = \{a^n b^n c^n : n \geq 0\}$.

Witness and split

Choose $w = a^p b^p c^p$. Every legal split has $y = a^k$, $1 \leq k \leq p$.

Pump up

With $i = 2$, $xy^2z = a^{p+k} b^p c^p$. The counts differ; hence L is not regular.

Example 3: palindromes

Language

$L = \{w \in \{0,1\}^* : w = w^R\}$, where w^R is the reversal of w.

Witness and split

Choose $w = 0^p 1 0^p$. Every legal split has $y = 0^k$, $1 \le k \le p$.

Pump up

$i = 2$ gives $0^{p+k} 1 0^p$. The blocks around the unique 1 have unequal lengths, so it is not a palindrome.

Example 4: copying a word

Language

$L = \{uu : u \in \{0,1\}^*\}$.

Witness and split

Choose $w = (0^p1)(0^p1)$. Every legal split has $y = 0^k$, $1 \leq k \leq p$.

Pump up

$i = 2$ gives $0^{p+k}10^p1$. If it were vv, each half would contain one 1 and end in 1. The two zero blocks would need equal lengths. Contradiction.

Example 5: unary powers of two

Language

L = {a^(2ⁿ) : n ≥ 0}, where the length is 2 raised to n.

Witness and split

Choose m with $2^m > p$. Let w = a^(2^m). Any legal split has $y = a^k$, $1 \le k \le p$.

Pump up

For i = 2, $2^m < |xy^2z| = 2^m + k < 2^{m+1}$. No power of two lies between them; hence L is not regular.

Example 6: more zeros than ones

Language

$L = \{w \in \{0,1\}^* : \#_0(w) > \#_1(w)\}$.

Witness and split

Choose $w = 0^p 1^{p-1}$. Every legal split has $y = 0^k$, $1 \leq k \leq p$.

Pump down

$i = 0$ gives $p - k$ zeros and $p - 1$ ones. Since $p - k \leq p - 1$, the result is outside L.

Example 7: using closure properties

Language

L = {0,1}* minus $\{0^n1^n : n \geq 0\}$.

Assume regularity

If L were regular, its complement relative to {0,1}* would also be regular.

Contradiction

The complement is $\{0^n1^n : n \geq 0\}$, already proved nonregular in Lecture 31. Therefore L is nonregular.

When pumping does not prove regularity

Regular example

L = 0* has pumping length 1. For any nonempty w, choose x = ε, y = 0 and z = the remaining zeros.

All pumping values work

For every $i \geq 0$, xy^iz consists only of zeros, including possibly ε.

Logical limitation

The lemma is necessary, not sufficient. To prove regularity, construct a DFA, NFA or regular expression.

Diagnose an incorrect proof

Claim

For $L = \{0^n1^n\}$, choose $w = 0^p1^p$ and $y = 01$; pump it down.

Error

This split violates $|xy| \leq p$: the first 1 occurs at position $p + 1$. Also, one split cannot establish nonregularity.

Repair

Use arbitrary $y = 0^k$, $1 \leq k \leq p$, and pump with $i = 0$. The resulting counts differ.

Classroom exercises

Exercise A

Prove $L = \{a^n b^{3n} : n \geq 0\}$ is not regular.

Exercise B

Prove $L = \{0^n 1 0^n : n \geq 0\}$ is not regular.

Exercise C

Prove $L = \{a^n : n \text{ is a perfect square}\}$ is not regular. Choose a square far enough from the next square.

Solutions and recap

A

$w = a^p b^{3p}$, $y = a^k$. Pump $i = 0$: $p - k$ a's and $3p$ b's violate the ratio.

B

$w = 0^p 1 0^p$, $y = 0^k$. Pump $i = 2$: the two zero blocks become unequal.

C

$w = a^{(p^2)}$, $y = a^k$, $1 \le k \le p$. Pump $i = 2$: $p^2 < p^2 + k < (p+1)^2$. No square lies between.