Absolute, relative and percentage errors

NC 03 • Module 1: Foundations

Central question	How close is a numerical approximation to the exact value?
Lecture focus	Signed error, absolute error, relative error and percentage error.
Prerequisite	Basic arithmetic and floating-point representation (NC 02).

Learning outcomes

By the end of this lecture, students should be able to...

Define	Distinguish exact values, approximations and signed errors.
Calculate	Compute absolute, relative and percentage errors in worked problems.
Interpret	Choose an error measure, apply tolerances and recognize limitations near zero.

Exact and approximate values

Numerical computing often replaces an exact quantity with an approximation.

Exact value	x: the true mathematical value or an accepted reference value.
Approximate value	a: the value obtained by measurement, rounding or a numerical algorithm.
Example	For $x = \sqrt{2} \approx 1.41421356$, $a = 1.414$ is a finite decimal approximation.

Signed error

The sign reveals whether the approximation is too high or too low.

Definition	$e = x - a$ (this lecture uses true value minus approximate value).
Underestimate	$x = 10$, $a = 9.8$: $e = +0.2$. The approximation is too small.
Overestimate	$x = 10$, $a = 10.2$: $e = -0.2$. The approximation is too large.

Absolute error

Absolute error measures the magnitude of the discrepancy.

Formula	$E_a = \lvert x - a \rvert = \lvert e \rvert$.
Properties	$E_a \geq 0$; it has the same units as x. It is zero for an exact approximation.
Example	$x = 50$ cm, $a = 49.6$ cm: $E_a = \lvert 50 - 49.6 \rvert = 0.4$ cm.

Relative error

Relative error measures the discrepancy compared with the magnitude of the reference.

Formula	$E_r = \lvert x - a \rvert \,/\, \lvert x \rvert$, provided $x \neq 0$.
Properties	Dimensionless; useful when comparing quantities of different scales.
Example	$x = 50$ cm, $a = 49.6$ cm: $E_r = 0.4/50 = 0.008$.

Percentage error

Percentage error expresses absolute relative error as a percentage.

Formula	$E_p = 100 \times E_r$ %, where E_r is expressed as a fraction.
Example	$E_r = 0.008$ gives $E_p = 0.8\%$.
Interpretation	An error of 0.8% means the absolute discrepancy is 0.008 times the reference magnitude.

Worked example 1: all three errors

Given exact value x = 80 and approximate value a = 78.4.

Absolute error

$E_a = |80 - 78.4| = 1.6.$

Relative error

$E_r = 1.6/80 = 0.02.$

Percentage error

$E_p = 0.02 \times 100\% = 2\%$. Signed error is +1.6 (underestimate).

Worked example 2: approximating π

Use π ≈ 3.141592653589793 as the reference; take a = 3.14.

Absolute error

$E_a \approx |3.141592653589793 - 3.14| \approx 0.00159265.$

Relative error

$E_r \approx 0.00159265/3.14159265 \approx 0.000506957.$

Percentage error

$E_p \approx 0.0506957\%$. The approximation is below π.

Worked example 3: negative reference

Given $x = -20$ and $a = -19.5$. The denominator must be a magnitude.

Signed and absolute	$e = -20 - (-19.5) = -0.5$; $E_a = 0.5$.
Relative error	$E_r = 0.5/\lvert -20 \rvert = 0.025$.
Percentage error	$E_p = 2.5\%$. The approximation is greater than the exact value.

Comparing errors across scales

The same absolute error can have very different practical significance.

Case A	$x = 10$, $a = 9$: $E_a = 1$; $E_r = 0.1$; $E_p = 10\%$.
Case B	$x = 1000$, $a = 999$: $E_a = 1$; $E_r = 0.001$; $E_p = 0.1\%$.
Conclusion	Case B is relatively more accurate despite identical absolute errors.

Zero and near-zero references

Relative and percentage errors are not always suitable.

Exactly zero	$x = 0$, $a = 0.001$: $E_a = 0.001$; relative and percentage errors are undefined.
Near zero	$x = 0.0001$, $a = 0.0002$: $E_a = 0.0001$ but $E_p = 100\%$.
Practical choice	Use an absolute tolerance or a problem-specific scale near zero.

Error bounds and tolerances

A tolerance specifies the maximum acceptable discrepancy.

Absolute tolerance	Accept if $\lvert x - a \rvert \le \tau_a$. Example: $\tau_a = 0.01$ cm.
Relative tolerance	For $x \neq 0$, accept if $\lvert x - a \rvert / \lvert x \rvert \le \tau_r$.
Combined test	Accept if $\lvert x - a \rvert \le \tau_a + \tau_r \lvert x \rvert$, for nonnegative tolerances.

When the exact value is unknown

Successive approximations can estimate changes, not necessarily true error.

Absolute change	For iterates a_k and a_{k-1}: $\Delta_k = \lvert a_k - a_{k-1} \rvert$.
Relative change	If $a_k \neq 0$, $r_k = \Delta_k / \lvert a_k \rvert$; percentage change = $100 r_k\%$.
Caution	A small change does not by itself prove a small true error or convergence.

Worked example 4: successive iterates

An algorithm produces $a_1 = 1.5$, $a_2 = 1.416667$, $a_3 = 1.414216$.

Absolute change	$\Delta_3 = \lvert 1.414216 - 1.416667 \rvert = 0.002451$.
Relative change	$r_3 = 0.002451/1.414216 \approx 0.001733$; about 0.1733%.
True error check	If $x = \sqrt{2}$, $E_a \approx 0.000002438$. The change and true error differ.

Algorithm and Python implementation

Inputs: reference x and approximate value a (finite real numbers).

Step 1

```
ea = abs(x - a)   # absolute error
```

Step 2

```
er = ea / abs(x) if x != 0 else None
```

Step 3

```
ep = 100 * er if er is not None else None; return ea, er, ep.
```

Practice questions

Attempt each exercise before viewing the solutions.

Q1 For $x = 250$ and $a = 247.5$, calculate signed, absolute, relative and percentage errors.

Q2 For $x = -8$ and $a = -8.2$, find all four error measures.

Q3 For $x = 0$ and $a = 0.0003$, which error measures are defined?

Solutions and lecture recap

Next lecture: NC 04 — Round-off, Truncation and Propagation of Errors.

A1 $e = +2.5$; $E_a = 2.5$; $E_r = 0.01$; $E_p = 1\%$.

A2 $e = +0.2$; $E_a = 0.2$; $E_r = 0.025$; $E_p = 2.5\%$.

A3 $e = -0.0003$; $E_a = 0.0003$. Relative and percentage errors are undefined.