Taylor series, stability and convergence

NC 05 • Module 1: Foundations

Central question	How can we approximate functions and judge whether numerical algorithms are reliable?
Lecture focus	Taylor polynomials, remainder bounds, convergence rates and numerical stability.
Prerequisite	Differentiation, floating-point arithmetic and error analysis (NC 02–04).

Learning outcomes

By the end of this lecture, students should be able to...

Derive	Construct Taylor and Maclaurin polynomials using derivatives.
Estimate	Calculate truncation errors and use Taylor remainder bounds.
Evaluate	Distinguish convergence, conditioning and numerical stability.

Taylor series: the central idea

Approximate a smooth function near a known expansion point a.

Constant term	f(a) matches the function value at the expansion point.
Linear term	$f'(a)(x - a)$ matches the slope.
Higher terms	$f''(a)(x - a)^2/2!$ and later terms capture curvature and higher derivatives.

Taylor polynomial of degree n

Assume that f has sufficiently many continuous derivatives near a.

General formula	$P_n(x) = \Sigma[k=0\ldots n]\ f^{(k)}(a)(x-a)^k/k!$.
Taylor theorem	$f(x) = P_n(x) + R_n(x)$, where R_n is the remainder.
Infinite series	The Taylor series equals $f(x)$ wherever $R_n(x) \to 0$ as $n \to \infty$.

Maclaurin series

A Maclaurin series is a Taylor series centered at a = 0.

Exponential	$e^x = 1 + x + x^2/2! + x^3/3! + \dots$ (all real x).
Sine	$\sin x = x - x^3/3! + x^5/5! - \dots$ (x in radians).
Cosine	$\cos x = 1 - x^2/2! + x^4/4! - \dots$ (x in radians).

Worked example: exponential

Approximate e^0.2 with a cubic Maclaurin polynomial.

Substitute $P_3(0.2) = 1 + 0.2 + 0.2^2/2 + 0.2^3/6$.

Evaluate $P_3(0.2) = 1.221333333$; reference ≈ 1.221402758.

Absolute error $|e^{0.2} - P_3(0.2)| \approx 0.000069425$.

Taylor remainder: Lagrange form

If f has n+1 continuous derivatives on the interval joining a and x.

Remainder	$R_n(x) = f^{(n+1)}(\xi)(x - a)^{n+1}/(n+1)!$ for some ξ between a and x.
Error bound	If $\lvert f^{(n+1)}(t)\rvert \le M$ throughout that interval, $\lvert R_n(x)\rvert \le M\lvert x - a\rvert^{n+1}/(n+1)!$.
Example	For e^0.2 and n = 3, bound = e^0.2 × $0.2^4/24 \approx 0.000081427$.

Worked example: sine

Approximate sin(0.3) in radians using terms through degree 3.

Polynomial	$P_3(0.3) = 0.3 - 0.3^3/6 = 0.2955$.
Reference	$\sin(0.3) \approx 0.295520207$; absolute error ≈ 0.000020207.
Sharper bound	$P_4 = P_3$ because the degree-4 coefficient is zero; $\lvert R_4 \rvert \le 0.3^5/120 = 0.00002025$.

Expansion around a nonzero point

Approximate ln(1.1) by expanding ln x around a = 1.

Derivatives

$f(1)=0$, $f'(1)=1$, $f''(1)=-1$, $f'''(1)=2$.

Substitute

$P_3(1.1) = 0.1 - 0.1^2/2 + 0.1^3/3 = 0.095333333$.

Compare

$\ln(1.1) \approx 0.095310180$; absolute error ≈ 0.000023154.

Convergence of a sequence

A sequence of approximations converges to L if its error tends to zero.

Definition	$\lim_{n\to\infty} x_n = L$, equivalently $\lvert x_n - L\rvert \to 0$.
Convergent example	$x_n = 1 + 1/2^n$ approaches 1: 1.5, 1.25, 1.125, …
Nonconvergent example	$x_n = (-1)^n$ alternates between -1 and 1; it has no single limit.

Convergence of Taylor series

A Taylor series may represent a function only within a limited interval.

Geometric series	$1/(1 - x) = 1 + x + x^2 + \ldots$ for $\lvert x \rvert < 1$.
At x = 0.5	Partial sums 1, 1.5, 1.75, 1.875, … converge to 2.
At x = 2	Partial sums 1, 3, 7, 15, … diverge; the series is not valid there.

Rate of iterative convergence

Let $e_n = |x_n - L|$, with nonzero errors sufficiently near the limit.

Linear	$e_{n+1}/e_n \to C$ with $0 < C < 1$; for example $e_n = 2^{-n}$.
Quadratic	$e_{n+1}/e_n^2 \to C > 0$; for example $e_0 = 0.1$ and $e_{n+1} = e_n^2$.
Interpretation	Quadratic convergence can roughly double the number of correct digits per step near the solution.

Numerical stability

Stability concerns how perturbations and rounding errors affect an algorithm.

Stable behavior	Small errors remain controlled under the relevant problem and algorithm assumptions.
Unstable behavior	Errors may grow excessively as operations or iterations continue.
Important distinction	An algorithm may converge in exact arithmetic but behave poorly in finite precision.

Worked example: error amplification

Consider the recurrence $z_{n+1} = c\, z_n$, with initial error $\Delta z_0 = 0.001$.

Error recurrence	$\Delta z_n = c^n \Delta z_0$ in exact arithmetic for an initial perturbation.
c = 0.5	After 10 steps, $\lvert\Delta z_{10}\rvert = 0.001/1024 \approx 9.77 \times 10^{-7}$.
c = 2	After 10 steps, $\lvert\Delta z_{10}\rvert = 0.001 \times 1024 = 1.024$; error grows.

Conditioning versus stability

Conditioning belongs to the mathematical problem; stability belongs to the algorithm.

Conditioning	For differentiable scalar f, relative condition number $\kappa = \lvert x f'(x)/f(x) \rvert$ when x and f(x) are nonzero.
Example	$f(x)=x-1$ at $x=1.001$: $\kappa=1001$, so relative input errors can be strongly amplified.
Stable reformulation	For large positive x, evaluate $\sqrt{(x+1)} - \sqrt{x}$ as $1/[\sqrt{(x+1)} + \sqrt{x}]$ to avoid cancellation.

Python: compute Taylor terms

Use a recurrence to avoid explicitly computing powers and factorials.

Initialize	x = 0.2; term = total = 1.0
Iterate	for k in range(1, 7): term *= x/k; total += term
Check	Compare total with math.exp(x): $P_6 \approx 1.22140275556$; error $\approx 2.61 \times 10^{-9}$.

Practice questions

Try these before checking the final slide.

Q1

Find the degree-2 Maclaurin approximation to $\cos(0.2)$ and an absolute error bound.

Q2

For $e_{n+1} = 0.25e_n$ and $e_0 = 0.04$, compute e_3 and identify the convergence rate.

Q3

For $z_{n+1} = 3z_n$, find the amplification of an initial error after five steps.

Solutions and module recap

Module 1 complete. Next: NC 06 — Graphical Method and Root Isolation.

A1	$P_2 = 1 - 0.2^2/2 = 0.98$; $P_3 = P_2$, so $\lvert R_3 \rvert \le 0.2^4/24 \approx 0.000066667$.
A2 / A3	$e_3 = 0.04(0.25)^3 = 0.000625$; linear convergence. Amplification = $3^5 = 243$.
Takeaway	Taylor polynomials approximate functions; convergence measures limiting behavior; stability controls error growth.