Graphical method and root isolation

NC 06 • Module 2: Solution of nonlinear equations

Central question	How can we locate solutions of f(x) = 0 before applying an iterative numerical method?
Lecture focus	Graphical estimates, sign changes, bracketing and systematic interval scanning.
Prerequisite	Functions, continuity, errors and convergence (NC 01–05).

Learning outcomes

By the end of this lecture, students should be able to...

Interpret	Identify real roots as intersections of a function with the horizontal axis.
Isolate	Use continuity and sign changes to construct intervals containing roots.
Evaluate	Recognize limitations of graphical scanning and select useful starting intervals.

Nonlinear equations

A root is a value r satisfying f(r) = 0.

Polynomial	$x^3 - x - 2 = 0$ is a nonlinear equation.
Transcendental	$\cos x - x = 0$ combines trigonometric and algebraic terms.
Numerical objective	Find an approximate real root to a specified tolerance when an exact expression is unavailable or inconvenient.

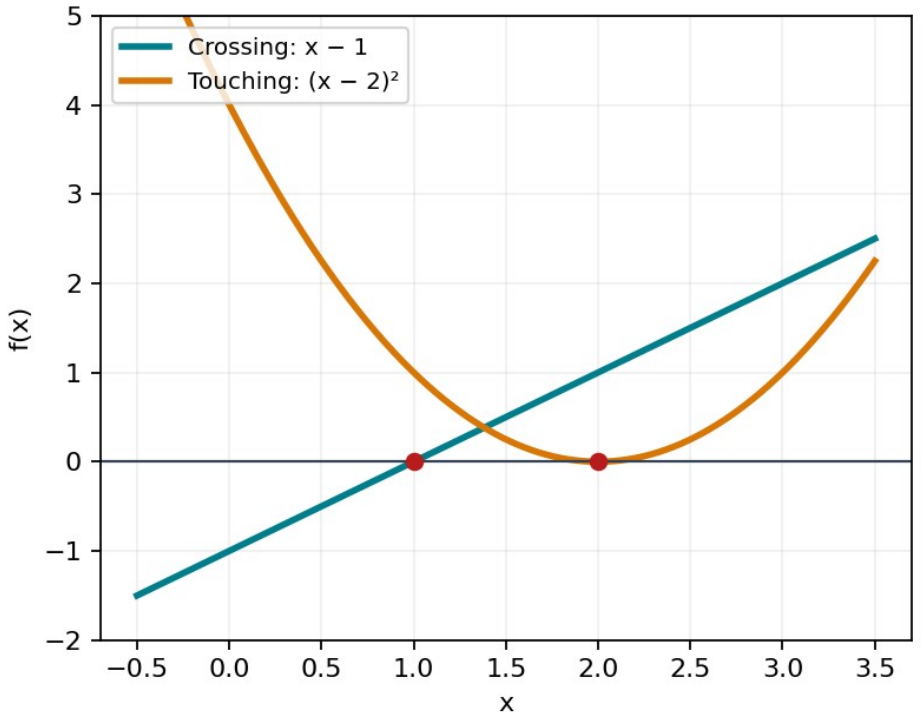
Graphical interpretation

Plot y = f(x) and look for intersections with y = 0.

Crossing — A simple root often crosses the x-axis, changing the sign of f.

Touching — An even-multiplicity root can touch the x-axis without changing sign.

Estimate — Read the approximate x-coordinate; graphical accuracy depends on scale and resolution.



Intermediate value theorem

Suppose f is continuous on the closed interval [a,b].

Sign-change condition	If $f(a)f(b) < 0$, then at least one root lies in (a,b).
Endpoint check	If $f(a) = 0$ or $f(b) = 0$, an endpoint is itself a root.
Limitation	A sign change guarantees existence, not uniqueness. Continuity is essential.

Root isolation versus bracketing

A bracket guarantees at least one root; isolation identifies exactly one.

Bracket	Find a and b with opposite function signs and continuity on [a,b].
Uniqueness	If f is strictly monotone on [a,b], there can be at most one root.
Sufficient test	A derivative strictly positive or strictly negative throughout the interval establishes strict monotonicity.

Example 1: cubic equation

Locate the positive root of $f(x) = x^3 - x - 2$.

Evaluate at x = 1

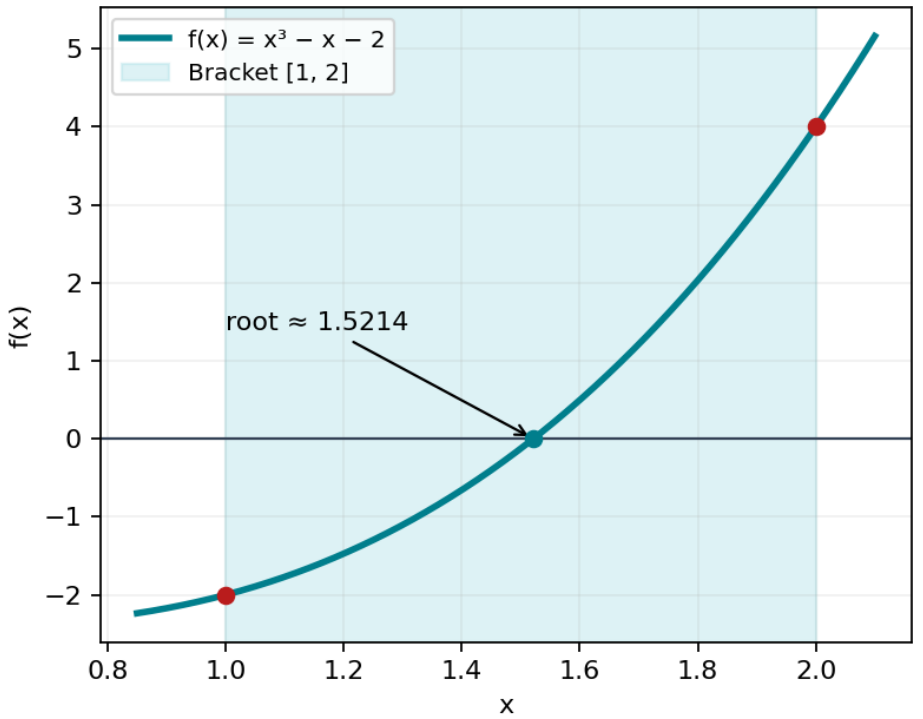
$f(1) = 1 - 1 - 2 = -2$.

Evaluate at x = 2

$f(2) = 8 - 2 - 2 = 4$.

Conclusion

Continuity and opposite signs imply a root in (1,2). Since $f'(x)=3x^2-1 > 0$ there, the root is unique.



Example 1: refine the bracket

Use smaller sampling intervals near the crossing.

First refinement	f(1.5) = −0.125; f(1.6) = 0.496. Bracket: [1.5,1.6].
Second refinement	f(1.52) = −0.008192; f(1.53) = 0.051577.
Estimate	Root lies in (1.52,1.53). Midpoint = 1.525 with absolute error at most 0.005.

Example 2: transcendental equation

Isolate a root of $f(x) = \cos x - x$, with x in radians.

Initial interval	$f(0) = 1$; $f(1) \approx -0.459698$. A root lies in $(0,1)$.
Refine	$f(0.73) \approx 0.015174$; $f(0.74) \approx -0.001531$.
Uniqueness	$f'(x) = -\sin x - 1 < 0$ on $[0,1]$. The bracket $[0.73,0.74]$ isolates one root.

Example 3: multiple roots

Scan $f(x) = (x+2)(x-1)(x-3)$.

Left root	$f(-2.5) = -9.625$; $f(-1.5) = 5.625$. Root bracket: $[-2.5, -1.5]$.
Middle root	$f(0.5) = 3.125$; $f(1.5) = -2.625$. Root bracket: $[0.5, 1.5]$.
Right root	$f(2.5) = -3.375$; $f(3.5) = 6.875$. Root bracket: $[2.5, 3.5]$.

When sign changes fail

Not every root is detected by a sign-change test.

Repeated root	$f(x) = (x-2)^2$ is nonnegative on both sides of $x = 2$.
Two roots in one cell	$f(x) = (x-1)(x-2)$ has $f(0) = f(3) = 2$; both roots are missed by this coarse interval.
Remedy	Use finer sampling, inspect minima and maxima, or analyze derivatives and known structure.

Discontinuity: false brackets

A sign change alone is not enough if continuity fails.

Counterexample	$f(x) = 1/x$ on $[-1,1]$ has $f(-1)=-1$ and $f(1)=1$.
Problem	The function is undefined at $x=0$ and never equals zero.
Lesson	Check the domain and split intervals at singularities before applying a root-bracketing method.

Systematic grid scanning

Choose a search interval [A,B] and a positive step h.

Sample	Evaluate f at $x_i = A + ih$, including B as the final point if necessary.
Detect	Record exact sample roots and consecutive finite values of opposite signs within continuous subintervals.
Refine	Decrease h or subdivide candidate intervals; verify uniqueness when isolation is required.

Algorithm: root bracketing

Input: continuous f on [A,B], integer N > 0.

Initialize	h = (B−A)/N; x = A; u = f(x). Record x if u = 0.
Repeat N times	Set y = next grid point; v = f(y). Record y if v = 0; otherwise record [x,y] if u and v have opposite signs.
Update and return	Set x = y and u = v. Return the recorded roots and brackets; do not claim all roots were found.

Python implementation

Example scan for $f(x) = x^3 - x - 2$ on [1,2].

Initialize

```
f = lambda x: x**3 - x - 2
a, b, n = 1.0, 2.0, 100
```

Scan

```
x = a
for i in range(1, n+1):
    y = a + (b-a)*i/n
    if f(x)*f(y) < 0: print(x, y)
    x = y
```

Result

Prints approximately 1.52, 1.53. This simplified example assumes no exact grid root; the general algorithm checks those separately.

Accuracy and practical limitations

Graphical and grid methods provide initial estimates, not guaranteed exhaustive solutions.

Midpoint estimate	For a verified bracket [a,b], m = (a+b)/2 has $\lvert m-r\rvert \le (b-a)/2$ for any root r in the bracket.
Resolution	Smaller h increases cost and may still miss closely spaced or touching roots.
Next step	Use a bracket with bisection or false position; use other methods when a suitable bracket is unavailable.

Practice questions

Attempt these before reviewing the solutions.

Q1

Find a bracket of width 1 for a positive root of $x^2 - 5 = 0$, and establish uniqueness.

Q2

Explain why $f(x) = (x-1)^2$ on $[0,2]$ has a root despite equal positive endpoint values.

Q3

Does $f(x) = 1/(x-2)$ have a root in $[1,3]$? Explain the sign change.

Solutions and recap

Next: NC 07 — Bisection Method.

A1	$f(2)=-1$, $f(3)=4$; $[2,3]$ brackets a root. $f'(x)=2x>0$ there, so exactly one root lies inside.
A2 / A3	The repeated root at 1 touches the axis. The reciprocal has no root: $x=2$ is a discontinuity, not a zero.
Takeaway	Check continuity, locate sign changes, verify uniqueness and refine the bracket before numerical iteration.