Fixed-Point Iteration Method

NC 09 • Module 2: Solution of nonlinear equations

Central question	How can a nonlinear equation be solved by repeatedly applying a function?
Lecture focus	Rearrangement, cobweb diagrams, convergence, error bounds and implementation.
Prerequisite	Root isolation, bisection and false position (NC 06–08).

Learning outcomes

By the end of this lecture, students should be able to...

Explain	Transform f(x)=0 into an equivalent fixed-point problem x=g(x).
Compute	Construct iteration tables and interpret cobweb diagrams.
Evaluate	Check convergence conditions, estimate errors and recognize divergence.

What is a fixed point?

A fixed point r of g satisfies g(r)=r.

Root equation	Solve f(x)=0 by selecting a rearrangement x=g(x) with equivalent solutions on the chosen domain.
Iteration	Start from x_0 and compute $x_{n+1}=g(x_n)$, for n=0,1,2,...
Limit	If x_n converges to r and g is continuous there, then r=g(r).

Geometric interpretation

A fixed point is the intersection of y=g(x) and y=x.

Start

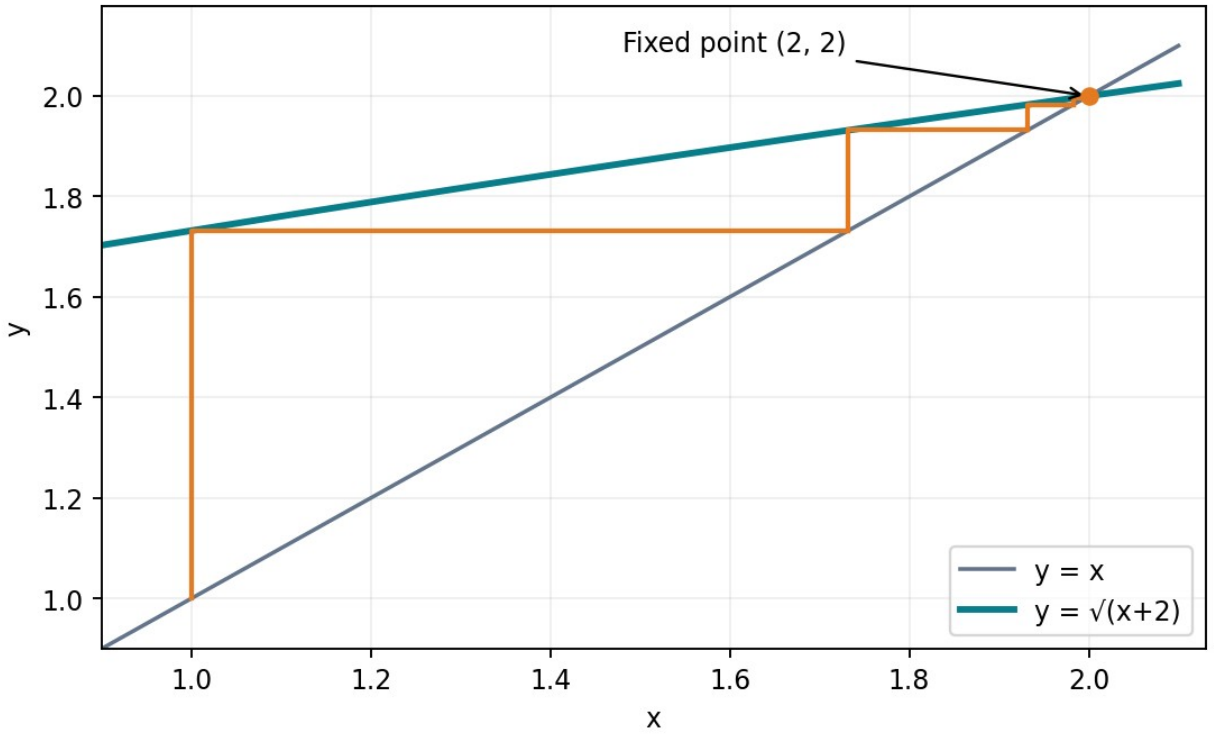
Choose x_0 on the horizontal axis.

Vertical step

Move vertically to the curve y=g(x).

Horizontal step

Move horizontally to y=x, then repeat. The resulting path is a cobweb.



Choosing the iteration function

Different rearrangements of the same equation can behave differently.

Equation	$x^2-x-2=0$ has roots -1 and 2.
Choice A	$g(x)=\sqrt{(x+2)}$, valid for the nonnegative root. $g'(2)=1/4$.
Choice B	$g(x)=x^2-2$ has $g'(2)=4$. Near 2 this iteration is repelling.

Contraction theorem

A sufficient global convergence condition on a closed interval [a,b].

Self-map

g is continuous and maps [a,b] into itself.

Contraction

If $|g(x)-g(y)| \le q|x-y|$ for all x,y in [a,b], with $0 \le q < 1$, g is a contraction.

Conclusion

There is exactly one fixed point in [a,b]; iteration converges to it from every starting point in that interval.

Local convergence test

Assume g is continuously differentiable near a fixed point r.

Attracting $|g'(r)| < 1$ implies convergence for sufficiently close initial values.

Repelling $|g'(r)| > 1$ implies nearby nonfixed iterates move away locally.

Borderline $|g'(r)| = 1$ is inconclusive; additional analysis is needed.

Example 1: polynomial equation

Solve $x^2-x-2=0$ for its positive root using $g(x)=\sqrt{(x+2)}$.

Initial guess	$x_0=1$. On $[1,2]$, g maps into $[\sqrt{3},2]$.
Derivative	$g'(x)=1/(2\sqrt{(x+2)}) \le 1/(2\sqrt{3}) < 0.289$.
First step	$x_1=\sqrt{3}\approx 1.732050808$; $x_2=\sqrt{(x_1+2)}\approx 1.931851653$.

Example 1: iteration table

Starting at $x_0=1$. Values are rounded only for display.

Iterations 1–2	1: x = 1.732050808; change = 0.732050808 2: x = 1.931851653; change = 0.199800845
Iterations 3–4	3: x = 1.982889723; change = 0.051038070 4: x = 1.995717846; change = 0.012828124
Iterations 5–6	5: x = 1.998929175; change = 0.003211328 6: x = 1.999732276; change = 0.000803101

Error bounds for a contraction

Let q be a verified contraction constant, and r the fixed point.

A priori

$$|x_n - r| \le q^n |x_1 - x_0| / (1-q).$$

A posteriori

$$|x_n - r| \le q |x_n - x_{n-1}| / (1-q), \; n \ge 1.$$

Example 1

Use q=0.289. At n=6 the posterior bound is about 0.0003264.

Example 2: transcendental equation

Solve $\cos(x)-x=0$ with angles in radians.

Rearrangement	$g(x)=\cos(x)$, $x_0=0.5$.
Self-map	On $[0,1]$, $\cos(x)$ lies in $[\cos(1),1]$, a subset of $[0,1]$.
Derivative	$\lvert g'(x)\rvert=\lvert\sin(x)\rvert \le \sin(1)<0.842$; convergence is guaranteed.

Example 2: iteration table

The estimates oscillate around r≈0.739085133.

Iterations 1–2	1: x = 0.877582562; change = 0.377582562 2: x = 0.639012494; change = 0.238570068
Iterations 3–4	3: x = 0.802685101; change = 0.163672607 4: x = 0.694778027; change = 0.107907074
Iterations 5–6	5: x = 0.768195831; change = 0.073417804 6: x = 0.719165446; change = 0.049030385

Divergence and oscillation

A rearrangement is not automatically a good iteration function.

Divergence	For $g(x)=x^2-2$, starting at 2.1 gives 2.41, 3.8081, then rapidly increasing values.
Oscillation	$g(x)=1-x$ has fixed point 0.5, but starting at 0.2 alternates 0.2, 0.8.
Lesson	Check the derivative and the iteration domain before computing.

Algorithm and stopping criteria

Input g, x_0, tolerance and maximum iteration count.

Compute	Evaluate y=g(x); verify y is finite and inside the intended domain.
Test	If a contraction constant q is known, stop when $q|y-x|/(1-q) \le$ tolerance.
Repeat	Otherwise update x=y. Use an iteration cap; small changes alone do not certify root accuracy.

Rate of convergence

Taylor expansion near a simple fixed point explains the local rate.

Error relation	$e_{n+1} = g'(r)e_n + O(e_n^2)$, for twice continuously differentiable g.
Linear	If $0 < \|g'(r)\| < 1$, the asymptotic error ratio is $\|g'(r)\|$.
Faster case	If $g'(r)=0$ and g'' is bounded nearby, convergence is at least quadratic locally.

Python implementation

Example with a known contraction constant q on an invariant interval.

Setup

```
def fixed_point(g, x, q, tol=1e-6, limit=1000):
    if not 0 <= q < 1 or tol <= 0: raise ValueError()
```

Iteration

```
    for _ in range(limit):
        y = g(x)
        if q*abs(y-x)/(1-q) <= tol: return y
        x = y
```

Finish / test

```
    raise RuntimeError("iteration limit")
# fixed_point(math.cos, 0.5, 0.842)
```

Practice questions

Attempt the questions before viewing the solutions.

Q1 For $g(x)=(x+3)/2$, compute x_1, x_2, x_3 from $x_0=0$.

Q2 Find the contraction constant for $g(x)=(x+3)/2$ and its fixed point.

Q3 Why is $g(x)=1-x$ not a convergent iteration from $x_0=0.2$?

Solutions and recap

Next: NC 10 — Newton-Raphson Method.

A1	$x_1=1.5$, $x_2=2.25$, $x_3=2.625$.
A2	$q=0.5$; solving $r=(r+3)/2$ gives $r=3$.
A3 / Takeaway	The iterates alternate 0.2 and 0.8; $\|g'\|=1$. Choose a contraction on an invariant interval.