Comparison and convergence

NC 12 • Module 2: Solution of nonlinear equations

Central question	Which root-finding method should we choose, and why?
Lecture focus	Compare reliability, convergence order, computational cost and stopping tests.
Prerequisite	Root isolation, bisection, false position, fixed-point, Newton and secant (NC 06–11).

Learning outcomes

By the end of this lecture, students should be able to...

Compare	Distinguish bracketing and open methods; identify their assumptions.
Compute	Compare five methods on a common equation using error and residual.
Evaluate	Estimate convergence order and select a method appropriate to the problem.

Two families of methods

The goal is to find r such that f(r) = 0.

Bracketing	Bisection and false position preserve a sign-changing interval for continuous f.
Open methods	Fixed-point, Newton and secant use estimates rather than a guaranteed bracket.
Hybrid methods	Combine a fast open step with a safe bracketing fallback.

Convergence: the definition

Let $e_n = x_n - r$, where r is the exact root.

Convergent sequence	$x_n \to r$, equivalently $\lvert e_n \rvert \to 0$.
Order p	If $\lvert e_{n+1} \rvert \,/\, \lvert e_n \rvert^p \to C$, with $0 < C < \infty$, the order is p.
Linear case	For $p = 1$, require $0 < C < 1$. Smaller C means faster asymptotic decay.

Linear, superlinear and quadratic

Convergence order describes behavior sufficiently close to the root.

Linear (p = 1)	Error shrinks by approximately a constant factor each iteration.
Superlinear (p > 1)	Error ratio $\lvert e_{n+1} \rvert$ / $\lvert e_n \rvert$ tends to zero. Secant has generic order ≈ 1.618.
Quadratic (p = 2)	Error is approximately proportional to its previous square. Newton typically has this order at a simple root.

Bisection and false position

Both begin with a continuous function and a sign-changing bracket.

Bisection	$c = (a+b)/2$. After k halvings, bracket width is $(b_0-a_0)/2^k$.
False position	$c = b - f(b)(b-a)/(f(b)-f(a))$. Replace the endpoint with the same sign.
Trade-off	Bisection guarantees geometric width reduction; false position may stagnate at one endpoint.

Fixed-point iteration

Rewrite the equation as x = g(x).

Update	$x_{n+1} = g(x_n)$. Different rearrangements can behave very differently.
Guarantee	If g maps a closed interval into itself and $\|g'\| \le q < 1$ there, iteration converges uniquely.
Local rate	Typically linear with $C = \|g'(r)\|$ if $0 < \|g'(r)\| < 1$.

Newton and secant

Fast local methods; neither preserves a bracket automatically.

Newton	$x_{n+1} = x_n - f(x_n)/f'(x_n)$. Needs a derivative; typically quadratic at a simple root.
Secant	Replace the derivative with the divided difference of the two latest estimates.
Limitations	Poor initial estimates, small denominators or multiple roots can reduce speed or cause failure.

Common benchmark: $x^2 - 2 = 0$

Exact positive root $r = \sqrt{2} \approx 1.414213562373095$.

Bracket	Use [1, 2] for bisection and false position; $f(1) = -1$, $f(2) = 2$.
Open methods	Newton and fixed-point start at 2; secant starts at 1 and 2.
Fixed-point choice	$g(x) = (x+2)/(x+1)$. On [1,2], $\lvert g'(x)\rvert = 1/(x+1)^2 \le 1/4$.

Benchmark: bracketing estimates

Five successive updates (left to right). Rounded for display only.

Bisection	1.50000000 | 1.25000000 | 1.37500000 | 1.43750000 | 1.40625000
False position	1.33333333 | 1.40000000 | 1.41176471 | 1.41379310 | 1.41414141
Observation	False position is faster here, but bisection has a predictable bracket-width bound.

Benchmark: open-method estimates

Five successive updates (left to right). Rounded for display only.

Fixed-point	1.33333333 | 1.42857143 | 1.41176471 | 1.41463415 | 1.41414141
Newton	1.50000000 | 1.41666667 | 1.41421569 | 1.41421356 | 1.41421356
Secant	1.33333333 | 1.40000000 | 1.41463415 | 1.41421144 | 1.41421356

Benchmark: absolute errors

Each row contains $|x_k - \sqrt{2}|$ for updates 1–5.

Bisection / false position	8.58e-02	1.64e-01	3.92e-02	2.33e-02	7.96e-03
	8.09e-02	1.42e-02	2.45e-03	4.20e-04	7.21e-05
Fixed-point / Newton	8.09e-02	1.44e-02	2.45e-03	4.21e-04	7.21e-05
	8.58e-02	2.45e-03	2.12e-06	1.59e-12	2.22e-16
Secant	8.09e-02	1.42e-02	4.21e-04	2.12e-06	3.16e-10

Fair comparison of work

Iterations alone are not a fair measure of computational cost.

Function evaluations	Bisection, false position and secant need one new f evaluation per update when values are cached.
Derivative evaluations	Newton typically uses one f and one f′ per update; derivative cost varies.
Other costs	Fixed-point uses one g evaluation per update; compare runtime, accuracy and reliability together.

Stopping criteria and error bounds

Use a tolerance appropriate to the scale of the problem.

Bracket certificate	For a midpoint of a valid bracket, $\lvert c-r\rvert \le (b-a)/2$ for a root r in the bracket.
Open-method tests	Check both $\lvert x_{n+1}-x_n\rvert \le \text{atol} + \text{rtol}\lvert x_{n+1}\rvert$ and a scaled residual tolerance.
Safeguards	Limit iterations; check nonfinite values and small denominators. A small step alone does not certify accuracy.

Estimating observed order

When the exact root is known, use three consecutive nonzero errors.

Estimator	$p \approx \ln(\lvert e_{n+1}/e_n \rvert) / \ln(\lvert e_n/e_{n-1} \rvert)$, when the denominator is nonzero.
Worked example	Errors 0.1, 0.01, 0.0001 give $p \approx \ln(0.01)/\ln(0.1) = 2$.
Caution	Use asymptotic iterations before round-off dominates; bisection midpoint errors can fluctuate.

Method selection and hybrid strategy

Choose reliability first, then speed.

Known sign change	Use bisection for a certified bracket; consider a safeguarded faster method.
Derivative available	Use Newton near a simple root; otherwise secant may be preferable.
Hybrid rule	Accept an open step only if it is safe and makes adequate progress; otherwise bisect the bracket.

Practice questions

Attempt these before viewing the solutions.

Q1

For an initial bracket of width 2, how many halvings ensure its width is $\leq 10^{-3}$?

Q2

Classify the convergence order when errors follow $|e_{n+1}| \approx 3|e_n|^2$.

Q3

Why might Newton be unsuitable when f' is nearly zero? Which method offers a safe fallback if a bracket exists?

Solutions and module recap

Module 2 complete • Next: NC 13 — Systems of Linear Equations.

A1 $2/2^k \le 10^{-3}$ implies $k \ge \log_2(2000)$; 11 halvings suffice.

A2 Quadratic convergence: $p = 2$, $C = 3$.

A3 A near-zero derivative may create a huge step. Bisection is a safe fallback for a valid continuous sign-changing bracket.