Introduction to systems of linear equations

NC 13 • Module 3: Solution of systems of linear equations

Central question	How can we find several unknowns that satisfy several equations simultaneously?
Lecture focus	Matrix representation, solution geometry, consistency, rank and residuals.
Prerequisite	Basic algebra, matrix operations and numerical error analysis (NC 01–05).

Learning outcomes

By the end of this lecture, students should be able to...

Represent	Convert a system of linear equations into Ax = b and an augmented matrix.
Classify	Recognize unique, infinite and inconsistent systems using rank and geometry.
Solve and verify	Solve small systems by elimination and evaluate the residual of an approximate solution.

What is a linear system?

A linear system contains m equations in n unknowns.

General form

$$a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n = b_i, \quad i = 1, \ldots, m.$$

Linear terms

Unknowns occur only to the first power; coefficients and right-hand sides are known.

Not linear

$x^2 + y = 4$ and $xy = 2$ are nonlinear because of powers or products of unknowns.

Matrix representation

The compact notation is Ax = b.

A: coefficient matrix	A has m rows and n columns; entry a_{ij} multiplies unknown x_j in equation i.
x: unknown vector	$x = (x_1, x_2, \ldots, x_n)^T$ has n components.
b: right-hand side	$b = (b_1, b_2, \ldots, b_m)^T$ has m components. Ax and b have matching dimensions.

Example 1 — build the matrix

Consider $2x + y = 5$ and $x - y = 1$.

Coefficient matrix — $A = [[2, 1], [1, -1]]$

Vectors — $x = [x, y]^T$; $b = [5, 1]^T$.

Augmented matrix — $[A \mid b] = [[2, 1 \mid 5], [1, -1 \mid 1]]$. The vertical bar separates coefficients from constants.

Geometric interpretation

Each equation in two unknowns describes a straight line (unless degenerate).

Unique solution	Two nonparallel lines intersect at exactly one point.
No solution	Distinct parallel lines never intersect: the equations contradict each other.
Infinitely many	Coincident lines share every point; at least one variable is free.

Example 1 — solve by elimination

$2x + y = 5;\quad x - y = 1.$

Step 1: eliminate y	Add the two equations: $3x = 6$.
Step 2: substitute	$x = 2$. From $x - y = 1$, obtain $y = 1$.
Step 3: verify	$2(2) + 1 = 5$ and $2 - 1 = 1$. Hence the unique solution is $(2, 1)$.

Three types of systems

Compare equations with the same or different slopes.

Unique	$x + y = 3$, $x - y = 1 \rightarrow (2, 1)$.
Infinite	$x + y = 3$, $2x + 2y = 6 \rightarrow (t, 3 - t)$ for any real t.
Inconsistent	$x + y = 3$, $2x + 2y = 7 \rightarrow 0 = 1$ after subtracting twice the first equation.

Elementary row operations

These operations preserve the solution set of an augmented matrix.

Swap	$R_i \leftrightarrow R_j$: exchange two equations.
Scale	$R_i \leftarrow cR_i$, $c \neq 0$: multiply an equation by a nonzero constant.
Replace	$R_i \leftarrow R_i + cR_j$ $(i \neq j)$: add a multiple of another equation.

Example 2 — three unknowns

$x + y + z = 6;\quad 2x - y + z = 3;\quad x + 2y - z = 2.$

Initial rows	$[1, 1, 1 \mid 6]$; $[2, -1, 1 \mid 3]$; $[1, 2, -1 \mid 2]$.
Eliminate x	$R_2 \leftarrow R_2 - 2R_1$ gives $[0, -3, -1 \mid -9]$. $R_3 \leftarrow R_3 - R_1$ gives $[0, 1, -2 \mid -4]$.
Eliminate y	$R_2 \leftarrow R_2 + 3R_3$ gives $[0, 0, -7 \mid -21]$. Thus $z = 3$.

Example 2 — back substitution

Use the reduced equations to recover the remaining unknowns.

Find y	$y - 2z = -4 \rightarrow y - 6 = -4 \rightarrow y = 2.$
Find x	$x + y + z = 6 \rightarrow x = 6 - 2 - 3 = 1.$
Verify	$(1, 2, 3)$: $1 + 2 + 3 = 6$; $2 - 2 + 3 = 3$; $1 + 4 - 3 = 2$.

Rank and consistency

Rank counts independent rows or columns; compute it using pivots.

Unique solution	rank(A) = rank([A | b]) = n.
Infinite solutions	rank(A) = rank([A | b]) < n; n − rank(A) free variables.
No solution	rank(A) < rank([A | b]); a contradictory row [0 … 0 | c], c ≠ 0, appears.

Square systems and determinant

For a square n × n matrix, the determinant tests invertibility.

Nonsingular	$\det(A) \neq 0$ → unique solution for every b; A^{-1} exists.
Singular	$\det(A) = 0$ → no unique solution; use rank to distinguish no solution from infinitely many.
Example	For $A = [[2, 1], [1, -1]]$, $\det(A) = -2 - 1 = -3$, so A is nonsingular.

Direct versus iterative methods

Numerical algorithms are selected according to matrix structure and cost.

Direct methods	Gaussian elimination, Gauss-Jordan and LU factorization; finite arithmetic steps in exact arithmetic.
Iterative methods	Jacobi and Gauss-Seidel improve an initial guess; convergence requires suitable conditions.
Practical choice	Dense small systems often favor direct methods; large sparse systems may favor iterative methods.

Residual and numerical accuracy

For an approximate solution $\hat{x}$, define $r = b - A\hat{x}$.

Example	$A = [[2, 1], [1, -1]]$, $b = [5, 1]^T$, $\hat{x} = [1.99, 1.02]^T$.
Compute	$A\hat{x} = [5.00, 0.97]^T$; $r = [0.00, 0.03]^T$; $\|r\|_\infty = 0.03$.
Interpret	A small residual does not always imply a small solution error, especially for ill-conditioned matrices.

Python implementation

Use a numerical solver rather than explicitly computing the inverse.

Set up

```
import numpy as np
A = np.array([[2., 1.], [1., -1.]])
b = np.array([5., 1.])
```

Solve and check

```
x = np.linalg.solve(A, b)
r = b - A @ x
print(x, np.linalg.norm(r, np.inf))
```

Expected output

[2. 1.] 0.0 (up to floating-point rounding). A singular matrix raises a LinAlgError.

Practice questions

Attempt these before reading the answers.

Q1

Write the augmented matrix for $3x + 2y = 7$, $x - y = 1$; solve the system.

Q2

Classify $x + 2y = 4$ and $2x + 4y = 8$ using rank.

Q3

For $x + y = 3$, $x - y = 1$ and $\hat{x} = (2.1, 0.9)$, calculate the residual $b - A\hat{x}$.

Solutions and recap

Next lecture: NC 14 — Gaussian Elimination Method.

Q1

[3, 2 | 7]; [1, −1 | 1]. Substitute x = 1 + y: 5y = 4, so y = 0.8, x = 1.8.

Q2

rank(A) = rank([A | b]) = 1 < 2; infinitely many solutions.

Q3

$A\hat{x} = [3, 1.2]^T$; $r = [0, -0.2]^T$. Key idea: represent, classify, solve and verify.