Gaussian Elimination Method

NC 14 • Module 3: Solution of systems of linear equations

Central question	How can a dense linear system be reduced to a triangular system?
Method	Forward elimination followed by back substitution.
Prerequisite	Matrix representation and elementary row operations (NC 13).

Learning outcomes

By the end of this lecture, students should be able to...

Derive	Explain the elimination multiplier and the triangular reduction.
Compute	Solve two- and three-variable systems using augmented matrices.
Evaluate	Recognize zero pivots, check residuals and estimate computational cost.

The two phases

Start with a square system Ax = b.

Forward elimination	Remove entries below the diagonal using row replacements.
Triangular system	Obtain Ux = c, where U is upper triangular.
Back substitution	Solve the last equation first, then work upward to find all unknowns.

Elimination multiplier

At stage k, use the current diagonal entry a_{kk} as the pivot.

Multiplier	$m_{ik} = a_{ik} / a_{kk}$, for $i = k + 1, \ldots, n$.
Row operation	$R_i \leftarrow R_i - m_{ik} R_k$. Apply the same operation to the right-hand side.
Why it works	The new entry is $a_{ik} - (a_{ik} / a_{kk})a_{kk} = 0$. The solution set is unchanged.

Matrix reduction pattern

For a 3 × 3 system, eliminate one column at a time.

Initial matrix

$$[a_{11}\ a_{12}\ a_{13} \mid b_1];\ [a_{21}\ a_{22}\ a_{23} \mid b_2];\ [a_{31}\ a_{32}\ a_{33} \mid b_3]$$

After stage 1

$$[u_{11}\ u_{12}\ u_{13} \mid c_1];\ [0\ v_{22}\ v_{23} \mid d_2];\ [0\ v_{32}\ v_{33} \mid d_3]$$

After stage 2

$$[u_{11}\ u_{12}\ u_{13} \mid c_1];\ [0\ u_{22}\ u_{23} \mid c_2];\ [0\ 0\ u_{33} \mid c_3]$$

Back substitution formula

Assume all diagonal entries of U are nonzero.

Last unknown

$x_n = c_n / u_{nn}$.

General formula

$x_i = (c_i - \sum_{j=i+1}^{n} u_{ij}x_j) / u_{ii}, \quad i = n-1, \ldots, 1.$

Interpretation

Subtract contributions of already known variables, then divide by the diagonal.

Example 1 — two unknowns

Solve $2x + y = 5$; $4x + 3y = 11$.

Augmented matrix	[2, 1 | 5]; [4, 3 | 11]. Pivot = 2.
Forward elimination	$m_{21} = 4/2 = 2$. $R_2 \leftarrow R_2 - 2R_1$ gives [0, 1 | 1].
Back substitution	$y = 1$; $x = (5 - 1)/2 = 2$. Check: $4(2) + 3(1) = 11$.

Example 2 — three unknowns

Solve $x + y + z = 6$; $2x - y + z = 3$; $x + 2y - z = 2$.

Initial augmented matrix	$[1, 1, 1 \mid 6]$; $[2, -1, 1 \mid 3]$; $[1, 2, -1 \mid 2]$.
First multiplier	$m_{21} = 2/1 = 2$. $R_2 \leftarrow R_2 - 2R_1 = [0, -3, -1 \mid -9]$.
Second multiplier	$m_{31} = 1/1 = 1$. $R_3 \leftarrow R_3 - R_1 = [0, 1, -2 \mid -4]$.

Example 2 — second pivot

After stage 1: [1, 1, 1 | 6]; [0, −3, −1 | −9]; [0, 1, −2 | −4].

Pivot	$a_{22} = -3$; $m_{32} = 1/(-3) = -1/3$.
Eliminate y	$R_3 \leftarrow R_3 + R_2/3 = [0, 0, -7/3 \mid -7]$.
Upper triangular system	$x + y + z = 6$; $-3y - z = -9$; $(-7/3)z = -7$.

Example 2 — back substitution

Solve the triangular equations from bottom to top.

Find z

$z = (-7)/(-7/3) = 3$.

Find y

$-3y - 3 = -9 \rightarrow -3y = -6 \rightarrow y = 2$.

Find x

$x = 6 - 2 - 3 = 1$. Solution: $(1, 2, 3)$.

Verify the solution

Substitute into the original equations, not just the triangular system.

Equation 1	$1 + 2 + 3 = 6$.
Equation 2	$2(1) - 2 + 3 = 3$.
Equation 3	$1 + 2(2) - 3 = 2$. Residual $r = b - Ax = (0, 0, 0)^T$.

Algorithm — forward elimination

Use a working augmented matrix M = [A | b].

Outer loop	For k = 1, …, n−1: check M[k,k] is nonzero.
Inner loop	For i = k+1, …, n: m = M[i,k]/M[k,k].
Update	For j = k, …, n+1: M[i,j] ← M[i,j] − m M[k,j].

Algorithm — back substitution

Use the triangular augmented matrix produced in phase 1.

Loop	For i = n, n−1, …, 1: check M[i,i] is nonzero.
Compute	$s = M[i,n+1] - \sum_{j=i+1}^{n} M[i,j]x_j$; $x_i = s/M[i,i]$.
Return	Return x and verify the residual using the original A and b.

Zero pivots and singular systems

Basic Gaussian elimination without pivoting may fail.

Zero pivot	For [0, 1 | 2]; [1, 1 | 3], swap the two rows before proceeding.
Contradictory row	[0, 0 | 5] means no solution. A zero row alone may indicate a free variable.
Small pivot	Dividing by a tiny pivot can amplify rounding errors. Partial pivoting is covered in NC 15.

Computational cost and accuracy

Gaussian elimination is a direct method for dense systems.

Forward elimination	Approximately $(2/3)n^3$ floating-point operations to reduce A; $O(n^3)$ overall.
Back substitution	$O(n^2)$ operations. Dense matrix storage requires $O(n^2)$ space.
Accuracy	Finite-precision arithmetic introduces rounding error. Check residuals; pivoting improves robustness.

Python implementation — no pivoting

The following function assumes a square float augmented matrix M, with nonzero safe pivots.

Forward elimination

```
for k in range(n-1):
    for i in range(k+1, n):
        M[i,k:] -= (M[i,k]/M[k,k])*M[k,k:]
```

Back substitution

```
x = np.zeros(n)
for i in range(n-1, -1, -1):
    x[i] = (M[i,n] - M[i,i+1:n] @ x[i+1:])/M[i,i]
```

Setup and checks

import numpy as np; M = np.column_stack((A,b)).astype(float); n = len(b). Check pivots before division; preserve A,b for residual.

Practice questions

Solve each problem before consulting the answers.

Q1 Solve $3x + 2y = 12$; $6x + 5y = 27$ using elimination.

Q2 Solve $x + y + z = 6$; $2x + 3y + z = 13$; $x + 2y + 3z = 14$.

Q3 Explain why the system $0x + y = 2$; $x + y = 3$ cannot begin with ordinary elimination without a row swap.

Solutions and recap

Next lecture: NC 15 — Gaussian Elimination with Pivoting.

Q1

$R_2 - 2R_1 \to [0, 1 \mid 3]$. $y = 3$; $x = 2$.

Q2

$R_2 - 2R_1 \to [0, 1, -1 \mid 1]$; $R_3 - R_1 \to [0, 1, 2 \mid 8]$. Subtract: $3z = 7$ $\to z = 7/3$, $y = 10/3$, $x = 1/3$.

Q3

The initial pivot is zero, so the multiplier is undefined. Swap rows; then $x = 1$, $y = 2$.