Gaussian Elimination with Pivoting

NC 15 • Module 3: Solution of systems of linear equations

Central question	How can Gaussian elimination avoid zero or dangerously small pivots?
Method	Select a suitable pivot, swap rows, eliminate and back substitute.
Prerequisite	Gaussian elimination without pivoting (NC 14).

Learning outcomes

By the end of this lecture, students should be able to...

Explain	Distinguish partial, scaled partial and complete pivoting.
Compute	Solve systems using partial pivoting and augmented matrices.
Evaluate	Recognize numerical instability, check residuals and interpret limitations.

Why pivoting is needed

Ordinary elimination divides by the current diagonal entry.

Zero pivot	A zero diagonal entry makes the elimination multiplier undefined.
Small pivot	A tiny denominator can produce a very large multiplier and amplify rounding errors.
Remedy	Exchange equations to place a larger available entry in the pivot position.

Partial pivoting rule

At stage k, search only the active rows in column k.

Select	Choose p such that $\lvert a_{pk} \rvert = \max\{\lvert a_{ik} \rvert : i = k, \ldots, n\}$.
Swap	Exchange entire rows R_k and R_p, including the right-hand side.
Eliminate	Use $m_{ik} = a_{ik} / a_{kk}$, then $R_i \leftarrow R_i - m_{ik} R_k$.

Why the rule helps

After partial pivoting, the pivot has largest magnitude in its active column.

Bound	$\lvert m_{ik} \rvert \le 1$ in exact arithmetic, provided the pivot is nonzero.
Benefit	Avoids large multipliers caused solely by choosing a small pivot.
Limitation	Does not guarantee small forward error for ill-conditioned systems or eliminate all element growth.

Example 1 — zero pivot

Solve 0x + 2y = 4; 3x + y = 5.

Initial matrix	[0, 2 | 4]; [3, 1 | 5]. Compare |0| and |3|.
Swap rows	Swap R_1 and R_2 gives [3, 1 | 5]; [0, 2 | 4].
Back substitution	y = 4/2 = 2; x = (5 − 2)/3 = 1. Both equations are satisfied.

Example 2 — small pivot

Solve 0.0001x + y = 1; x + y = 2. Use three significant digits.

Without pivoting	$m_{21} = 10000$. Updated row rounds to [0, −10000 | −10000].
Rounded solution	y = 1.00; x = (1 − 1.00)/0.0001 = 0. The true x is near 1.
With pivoting	Swap rows first. $m_{21} = 0.0001$; updated row rounds to [0, 1.00 | 1.00].

Small-pivot example — interpretation

The exact solution is $x = 1/0.9999$ and $y = 2 - x$.

Exact values	$x \approx 1.00010001$; $y \approx 0.99989999$.
Pivoted calculation	$y = 1.00$; $x = 2 - y = 1.00$ (three significant digits).
Key lesson	The unpivoted scheme loses information through rounding and cancellation, then divides by a tiny pivot.

Example 3 — first pivot

Solve x + y + z = 6; 3x + 2y + z = 10; 2x + 4y + z = 13.

Initial matrix	[1, 1, 1 | 6]; [3, 2, 1 | 10]; [2, 4, 1 | 13].
Select pivot	Largest magnitude in column 1 is 3. Swap R_1 and R_2.
After swap	[3, 2, 1 | 10]; [1, 1, 1 | 6]; [2, 4, 1 | 13].

Example 3 — first elimination

Use pivot 3 to eliminate the entries beneath it.

Row 2	$m_{21} = 1/3$; $R_2 \leftarrow R_2 - R_1/3 = [0, 1/3, 2/3 \mid 8/3]$.
Row 3	$m_{31} = 2/3$; $R_3 \leftarrow R_3 - 2R_1/3 = [0, 8/3, 1/3 \mid 19/3]$.
Next pivot	Compare $\lvert 1/3 \rvert$ and $\lvert 8/3 \rvert$. Swap R_2 and R_3.

Example 3 — second elimination

After the second swap, use 8/3 as the pivot.

Active rows	$R_2 = [0, 8/3, 1/3 \mid 19/3]$; $R_3 = [0, 1/3, 2/3 \mid 8/3]$.
Multiplier	$m_{32} = (1/3)/(8/3) = 1/8$.
Updated row	$R_3 \leftarrow R_3 - R_2/8 = [0, 0, 5/8 \mid 15/8]$.

Example 3 — back substitution

Solve the upper triangular system from the bottom upward.

Find z	z = (15/8)/(5/8) = 3.
Find y	(8/3)y + 1 = 19/3 → y = 2.
Find x	3x + 4 + 3 = 10 → x = 1. Check original equations: 6, 10, 13.

Algorithm — partial pivoting

Work on an augmented matrix M = [A | b].

For each column k	Find p = argmax |M[i,k]| for i = k, …, n. If the maximum is zero, stop: no unique solution.
Swap and eliminate	Swap rows k and p. For i > k, subtract (M[i,k]/M[k,k]) times row k from row i.
Finish	Check the last pivot, apply back substitution and verify against original A and b.

Other pivoting strategies

Partial pivoting is the standard choice for many dense systems.

Scaled partial	Use max $\lvert a_{ik}\rvert / s_i$, where $s_i = \max_j \lvert a_{ij}\rvert$ from the original row; swap scales with rows.
Complete	Search the active submatrix for largest $\lvert a_{ij}\rvert$. Swap rows and columns.
Trade-off	Complete pivoting needs more comparisons and requires tracking the variable permutation.

Complexity and numerical accuracy

Pivoting adds row searches and swaps to ordinary elimination.

Cost	Partial pivoting adds $O(n^2)$ comparisons; total dense solve remains $O(n^3)$.
Residual	Compute $r = b - A\hat{x}$ using original A and b. A small residual indicates a good equation fit.
Conditioning	A small residual need not imply a small solution error when A is ill-conditioned.

Python implementation

Assume NumPy imported as np; M = np.column_stack((A,b)).astype(float); n = len(b).

Pivot and eliminate

```
for k in range(n):
    p = k + np.argmax(abs(M[k:,k]))
    if M[p,k] == 0: raise ValueError("Singular")
    M[[k,p]] = M[[p,k]]
    for i in range(k+1,n): M[i,k:] -= M[i,k]/M[k,k]*M[k,k:]
```

Back substitution

```
x = np.zeros(n)
for i in range(n-1,-1,-1):
    x[i] = (M[i,n] - M[i,i+1:n] @ x[i+1:])/M[i,i]
```

Verification

Check np.linalg.norm(b-A@x, np.inf). In practice use scale-aware pivot checks.

Practice questions

Solve each problem before consulting the answers.

Q1 Use partial pivoting to solve $x + 2y = 5$; $4x + 3y = 11$.

Q2 For active column entries -2, 5, -7, which entry is selected? Why?

Q3 Explain why a column swap in complete pivoting requires tracking variable order.

Solutions and recap

Next lecture: NC 16 — Gauss-Jordan Elimination Method.

Q1

Swap rows; m = 1/4. $R_2 \leftarrow R_2 - R_1/4$ gives [0, 5/4 | 9/4]. y = 9/5; x = 7/5.

Q2

Select −7 because its absolute value 7 is largest.

Q3

Column swaps reorder unknowns. Restore the original ordering after back substitution.