Gauss-Jordan Elimination Method

NC 16 • Module 3: Solution of systems of linear equations

Central question	Can we solve a linear system directly from its final augmented matrix?
Method	Normalize each pivot and eliminate all other entries in its column.
Prerequisite	Gaussian elimination and partial pivoting (NC 14–15).

Learning outcomes

By the end of this lecture, students should be able to...

Explain	Distinguish row echelon form from reduced row echelon form (RREF).
Compute	Solve linear systems and compute matrix inverses using row operations.
Evaluate	Recognize inconsistent and underdetermined systems; verify solutions.

From Gaussian to Gauss-Jordan

Both methods preserve the solution set through elementary row operations.

Gaussian	Reduce A to upper triangular form; then use back substitution.
Gauss-Jordan	Reduce the augmented matrix to RREF; pivot columns have zeros above and below their pivots.
Unique solution	For a nonsingular square system, [A | b] becomes [I | x].

Elementary row operations

Each operation is reversible and preserves the solution set.

Exchange	$R_i \leftrightarrow R_j$: exchange two entire rows.
Scale	$R_i \leftarrow cR_i$, with $c \neq 0$: multiply an entire row by a nonzero scalar.
Replace	$R_i \leftarrow R_i + cR_j$, $i \neq j$: add a multiple of another row.

Reduced row echelon form

RREF is unique for a given matrix.

Leading entries	Each nonzero row begins with 1; successive leading ones move strictly to the right.
Pivot columns	Every leading 1 is the only nonzero entry in its column.
Zero rows	All zero rows occur below the nonzero rows.

The Gauss-Jordan algorithm

For a square nonsingular system, work on $M = [A \mid b]$.

Choose	At column k, choose the largest available magnitude in rows k through n; swap into row k.
Normalize	Divide the entire pivot row by its nonzero pivot M[k,k].
Eliminate	For every $i \neq k$, set $R_i \leftarrow R_i - M[i,k]R_k$; repeat for all columns.

Example 1 — first pivot

Solve $2x + y = 5$; $x - y = 1$.

Initial matrix	$[2, 1 \mid 5]$; $[1, -1 \mid 1]$.
Normalize R_1	$R_1 \leftarrow R_1/2$: $[1, 1/2 \mid 5/2]$; $[1, -1 \mid 1]$.
Eliminate below	$R_2 \leftarrow R_2 - R_1$: $[1, 1/2 \mid 5/2]$; $[0, -3/2 \mid -3/2]$.

Example 1 — second pivot

Eliminate above the second pivot as well.

Normalize R_2 $R_2 \leftarrow -(2/3)R_2$: [1, 1/2 | 5/2]; [0, 1 | 1].

Eliminate above $R_1 \leftarrow R_1 - R_2/2$: [1, 0 | 2]; [0, 1 | 1].

Read and verify x = 2, y = 1. Then 2x + y = 5 and x − y = 1.

Example 2 — first pivot

Solve $x + y + z = 6$; $2x + 3y + z = 11$; $x - y + 2z = 5$.

Initial matrix

$[1, 1, 1 \mid 6]$; $[2, 3, 1 \mid 11]$; $[1, -1, 2 \mid 5]$.

Eliminate below

For this hand example use the nonzero first pivot 1. $R_2 \leftarrow R_2 - 2R_1$; $R_3 \leftarrow R_3 - R_1$.

Result

$[1, 1, 1 \mid 6]$; $[0, 1, -1 \mid -1]$; $[0, -2, 1 \mid -1]$.

Example 2 — second pivot

The second pivot already equals 1.

Eliminate above	$R_1 \leftarrow R_1 - R_2 = [1, 0, 2 \mid 7]$.
Eliminate below	$R_3 \leftarrow R_3 + 2R_2 = [0, 0, -1 \mid -3]$.
Result	$[1, 0, 2 \mid 7]$; $[0, 1, -1 \mid -1]$; $[0, 0, -1 \mid -3]$.

Example 2 — third pivot

Normalize the last pivot and eliminate both entries above it.

Normalize	$R_3 \leftarrow -R_3 = [0, 0, 1 \mid 3]$.
Eliminate	$R_1 \leftarrow R_1 - 2R_3$; $R_2 \leftarrow R_2 + R_3$.
Solution	[1, 0, 0 | 1]; [0, 1, 0 | 2]; [0, 0, 1 | 3]. Thus (x,y,z) = (1,2,3).

Systems without a unique solution

For general systems, skip a coefficient column if no available nonzero pivot exists.

Inconsistent	A row [0, 0 | 1] represents 0 = 1; no solution exists.
Infinitely many	A consistent system with a free variable has infinitely many solutions.
Example	[1, 2 | 4]; [0, 0 | 0] gives y = t, x = 4 − 2t for any real t.

Computing a matrix inverse

Apply the same operations to both blocks of [A | I].

Start	A = [[2, 1], [1, 1]]. Augment: [2, 1 | 1, 0]; [1, 1 | 0, 1].
First pivot	$R_1 \leftarrow R_1/2$; $R_2 \leftarrow R_2 - R_1$: [1, 1/2 | 1/2, 0]; [0, 1/2 | −1/2, 1].
Second pivot	$R_2 \leftarrow 2R_2$; $R_1 \leftarrow R_1 - R_2/2$: [1, 0 | 1, −1]; [0, 1 | −1, 2].

Inverse verification

If [A | I] reduces to [I | B], then $B = A^{-1}$.

Result	$A^{-1} = [[1, -1], [-1, 2]]$.
Check	$AA^{-1} = [[2-1, -2+2], [1-1, -1+2]] = I$.
Limitation	A singular square matrix cannot reduce to I and has no inverse. To solve Ax=b, prefer a direct solve to forming an inverse.

Cost and numerical considerations

Gauss-Jordan and Gaussian elimination both have cubic dense-system cost.

Complexity	Gauss-Jordan: $O(n^3)$ arithmetic and $O(n^2)$ storage; typically more work than Gaussian elimination for one right-hand side.
Pivoting	Use partial pivoting to avoid zero or unnecessarily small pivots. It cannot cure poor conditioning.
Verification	Compute $r = b - A\hat{x}$ with the original inputs. A small residual alone does not guarantee small forward error.

Python implementation

For a nonsingular square A, with NumPy imported as np.

Initialize

```
M = np.column_stack((A,b)).astype(float)
n = len(b)
```

Reduce

```
for k in range(n):
    p = k + np.argmax(abs(M[k:,k]))
    if M[p,k] == 0: raise ValueError("Singular")
    M[[k,p]] = M[[p,k]]; M[k] /= M[k,k]
    for i in range(n):
        if i != k: M[i] -= M[i,k]*M[k]
```

Return

x = M[:,n].copy(); check np.linalg.norm(b-A@x, np.inf). Use scale-aware checks in production.

Practice questions

Solve each question before consulting the answers.

Q1 Solve x + 2y = 7; 3x + y = 11 using Gauss-Jordan elimination.

Q2 Find the inverse of [[1, 2], [3, 7]] by augmenting with I.

Q3 Classify the system with reduced rows [1, 0 | 2]; [0, 0 | 1].

Solutions and recap

Next lecture: NC 17 — LU Decomposition Method.

Q1

$R_2 \leftarrow R_2 - 3R_1$: [0, −5 | −10]. Normalize R_2; eliminate above. y = 2, x = 3.

Q2

$R_2 \leftarrow R_2 - 3R_1$; $R_1 \leftarrow R_1 - 2R_2$. Inverse = [[7, −2], [−3, 1]].

Q3

Inconsistent: the final row requires 0 = 1. Remember: normalize pivots, eliminate above and below.