Diagonal Dominance, Conditioning and Convergence

Lecture 20 | Reliability of linear system solvers

Core idea	Distinguish the sensitivity of a problem from the convergence of an algorithm.
Key tools	Diagonal dominance, matrix norms, condition number and spectral radius.
Learning focus	Worked examples, error bounds, method selection and computational checks.

Learning outcomes

By the end of this lecture, students should be able to:

Identify	Test strict diagonal dominance and recognize its role in iterative methods.
Calculate	Compute matrix norms, condition numbers and spectral radii for small systems.
Interpret	Distinguish residual, forward error, conditioning and iterative convergence.

Three different questions

A linear system Ax = b can be examined from three perspectives.

Existence	Does A have an inverse? A nonsingular square matrix has a unique solution.
Conditioning	How sensitive is that solution to perturbations in A or b?
Convergence	Does a chosen iterative method approach the solution from its initial guess?

Diagonal dominance

Compare the magnitude of each diagonal entry with the rest of its row.

Weak dominance	$\lvert a_{ii}\rvert \geq \Sigma \lvert a_{ij}\rvert$, $j \neq i$, for every row i.
Strict dominance	$\lvert a_{ii}\rvert > \Sigma \lvert a_{ij}\rvert$, $j \neq i$, for every row i.
Consequence	Strict row diagonal dominance guarantees nonsingularity and convergence of both Jacobi and Gauss-Seidel.

Example 1 — test dominance

A = [[4, −1, 1], [2, 6, 1], [1, 1, 5]]

Row 1 $|4| = 4 > |-1| + |1| = 2$.

Row 2 $|6| = 6 > |2| + |1| = 3$.

Row 3 $|5| = 5 > |1| + |1| = 2$. All rows are strictly dominant.

Example 2 — row rearrangement

Solve x + 4y = 9; 5x + y = 7.

Original order	A = [[1, 4], [5, 1]]. Neither row is strictly dominant.
Exchange rows	5x + y = 7; x + 4y = 9. New A = [[5, 1], [1, 4]].
Result	5 > 1 and 4 > 1. Both methods now converge. Rearrangement is not always possible.

Matrix and vector norms

Norms measure magnitude and enable quantitative error bounds.

Vector infinity norm $\|x\|_\infty = \max_i |x_i|$. For $x = (-2, 3, 1)$, the norm is 3.

Matrix infinity norm $\|A\|_\infty = \max_i \sum_j |a_{ij}|$ (largest absolute row sum).

Example $A = [[4, 1], [2, 3]]$ has row sums 5 and 5, so $\|A\|_\infty = 5$.

Condition number

For nonsingular A, use the same induced norm for A and its inverse.

Definition	$\kappa(A) = \|A\| \, \|A^{-1}\| \geq 1$. A singular matrix has infinite condition number.
Interpretation	A large condition number indicates potentially large relative solution changes from small relative data changes.
Important distinction	Conditioning is a property of the problem; stability is a property of the algorithm.

Example 3 — condition number

Use the infinity norm for A = [[4, 1], [2, 3]].

Step 1 $\det(A) = 12 - 2 = 10$; $A^{-1} = (1/10)[[3, -1], [-2, 4]]$.

Step 2 $\|A\|_\infty = 5$; $\|A^{-1}\|_\infty = \max(0.4, 0.6) = 0.6$.

Step 3 $\kappa_\infty(A) = 5 \times 0.6 = 3$. This system has modest sensitivity in this norm.

Example 4 — ill-conditioned system

A = [[1, 1], [1, 1.001]], b = (2, 2.001).

Original solution	x = (1, 1). det(A) = 0.001; $\kappa_\infty(A) = 4004.001$.
Perturb the data	Change only b_2 from 2.001 to 2.002. Subtract the first equation from the second: 0.001y = 0.002.
New solution	y = 2, x = 0. A tiny change in b produces a large solution change.

Residual versus forward error

Let x^* denote the exact solution and $\hat{x}$ the computed solution.

Residual	$r = b - A\hat{x}$; it measures how closely the computed vector satisfies the equations.
Forward error	$e = x^* - \hat{x} = A^{-1}r$. For $b \neq 0$, $\|e\|/\|x^*\| \leq \kappa(A)\, \|r\|/\|b\|$.
Example 4	For $\hat{x} = (0, 2)$ with original b, $r = (0, -0.001)$, but $\|e\|_\infty = 1$.

Stationary iteration framework

Write A = D + L + U using actual lower and upper entries.

Jacobi	$x^{(k+1)} = B_J x^{(k)} + c_J$; $B_J = -D^{-1}(L + U)$.
Gauss-Seidel	$B_{GS} = -(D + L)^{-1}U$; $c_{GS} = (D + L)^{-1}b$.
Error recurrence	$e^{(k+1)} = Be^{(k)}$. Hence $e^{(k)} = B^k e^{(0)}$.

Spectral radius and convergence

ρ(B) is the maximum absolute value of the eigenvalues of B.

Exact criterion	The iteration converges to its unique fixed point for every starting vector iff $\rho(B) < 1$.
Sufficient norm test	If $\|B\| < 1$ for an induced norm, convergence follows since $\rho(B) \leq \|B\|$.
Interpretation	$\rho(B) > 1$ implies divergence for some initial errors. At $\rho(B) = 1$, convergence is not guaranteed.

Example 5 — compare convergence

A = [[4, 1], [2, 3]]; solve using Jacobi or Gauss-Seidel.

Jacobi	$B_j = [[0, -1/4], [-2/3, 0]]$. Eigenvalues $\pm\sqrt{(1/6)}$; $\rho \approx 0.408248$.
Gauss-Seidel	$B_{GS} = [[0, -1/4], [0, 1/6]]$. Eigenvalues 0 and 1/6; $\rho \approx 0.166667$.
Conclusion	Both converge. Gauss-Seidel has the smaller asymptotic convergence factor in this example.

Convergence is not dominance alone

Diagonal dominance is sufficient, not necessary.

Example	$A = [[1, 2], [0, 1]]$ is not strictly dominant in row 1.
Jacobi matrix	$B_j = [[0, -2], [0, 0]]$; $B_j^2 = 0$ and $\rho(B_j) = 0$.
Result	The iteration reaches the exact solution within two sweeps in exact arithmetic, from any initial vector.

Python diagnostic toolkit

Use NumPy; check for zero diagonal entries before forming iteration matrices.

Conditioning

```
import numpy as np
A = np.array([[4.,1.],[2.,3.]])
print(np.linalg.cond(A, np.inf))  # 3
```

Iteration matrices

```
D = np.diag(np.diag(A)); L = np.tril(A,-1); U = np.triu(A,1)
BJ = -np.linalg.solve(D,L+U); BG = -np.linalg.solve(D+L,U)
```

Spectral radii

```
for B in (BJ, BG):
    print(max(abs(np.linalg.eigvals(B))))
# 0.408248... and 0.166667...
```

Practice questions

Attempt these before viewing the solutions.

Q1 Is $A = [[3, -1], [2, 5]]$ strictly row diagonally dominant?

Q2 Calculate κ_∞ for $A = \mathrm{diag}(1, 0.01)$.

Q3 If an iteration matrix has eigenvalues 0.2 and -1.1, does it converge for every initial guess?

Solutions and module recap

Module 3 complete: direct solvers, iterative solvers and reliability.

Q1	Yes: 3 > 1 and 5 > 2. Jacobi and Gauss-Seidel converge.
Q2 / Q3	$\kappa_\infty = 1 \times 100 = 100$. For Q3, $\rho = 1.1 > 1$, so not for every initial guess.
Next lecture	NC_21 — Introduction to Interpolation and Extrapolation.