Finite Differences and Difference Tables

Lecture 22 • Numerical Interpolation

Purpose	Convert equally spaced tabulated values into differences that reveal patterns.
Core tools	Forward, backward and central differences; successive difference tables.
Connection	Prepare for Newton forward and backward interpolation.

Learning outcomes

By the end of this lecture, students can:

Define	Explain the step size and finite difference operators.
Compute	Construct forward and backward difference tables by hand.
Apply	Recognize polynomial degree and choose a suitable interpolation direction.

Equally spaced data

Let $x_i = x_0 + ih$, and $y_i = f(x_i)$.

Step size	$h = x_{i+1} - x_i$; h must be constant for the standard Newton forward/backward formulas.
Example	x = 0, 1, 2, 3, 4 gives h = 1.
Unequal spacing	Use divided differences instead (Lecture 25).

Forward difference operator

Subtract the current value from the next value.

First order	$\Delta y_i = y_{i+1} - y_i$
Second order	$\Delta^2 y_i = \Delta y_{i+1} - \Delta y_i = y_{i+2} - 2y_{i+1} + y_i$
Higher orders	$\Delta^k y_i = \Delta^{k-1} y_{i+1} - \Delta^{k-1} y_i$

Backward difference operator

Subtract the preceding value from the current value.

First order	$\nabla y_i = y_i - y_{i-1}$
Second order	$\nabla^2 y_i = \nabla y_i - \nabla y_{i-1} = y_i - 2y_{i-1} + y_{i-2}$
Higher orders	$\nabla^k y_i = \nabla^{k-1} y_i - \nabla^{k-1} y_{i-1}$

Central differences

Use values symmetrically placed around the target.

First order	$\delta f(x) = f(x + h/2) - f(x - h/2)$
Second order	$\delta^2 f(x) = f(x+h) - 2f(x) + f(x-h)$
Derivative link	$f'(x) \approx [f(x+h) - f(x-h)] / (2h)$, for smooth f.

Constructing a difference table

Each new column has one fewer valid entry.

Step 1	Write the equally spaced x values and their corresponding y values.
Step 2	Subtract adjacent y values (later minus earlier) to form first differences.
Step 3	Repeat on each difference column until only one entry remains.

Example 1 — Forward difference table

Given $f(x) = x^2$ at $x = 0, 1, 2, 3, 4$.

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0	0	1	2	0	0
1	1	3	2	0	
2	4	5	2		
3	9	7			
4	16				

The second differences are constant (2); higher differences vanish.

Example 1 — Step-by-step calculations

Compute first and second differences explicitly.

First differences	$1-0=1$; $4-1=3$; $9-4=5$; $16-9=7$.
Second differences	$3-1=2$; $5-3=2$; $7-5=2$.
Third differences	$2-2=0$; $2-2=0$. The fourth difference is also zero.

Example 1 — Backward difference table

Same data, now aligned at the ending index.

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
0	0				
1	1	1			
2	4	3	2		
3	9	5	2	0	
4	16	7	2	0	0

At x = 4: $\nabla y_4 = 7$, $\nabla^2 y_4 = 2$, $\nabla^3 y_4 = 0$.

Example 2 — Cubic data

Given $f(x) = x^3$, with $h = 1$.

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0	0	1	6	6	0
1	1	7	12	6	
2	8	19	18		
3	27	37			
4	64				

The third differences are constant (6).

Polynomial degree and differences

For a polynomial of degree m with leading coefficient a_m:

Degree rule	$\Delta^m y = m!\, a_m\, h^m$, a constant; $\Delta^{m+1} y = 0$.
Reason	Each subtraction $f(x+h) - f(x)$ lowers polynomial degree by one.
Caution	Finite or noisy samples may resemble a polynomial without proving the underlying function is polynomial.

Example 3 — Non-unit step size

$f(x) = 2x^2 + 1$, $h = 2$.

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
0	1	8	16	0
2	9	24	16	
4	33	40		
6	73			

Check: $2! \times 2 \times 2^2 = 16$, matching the constant second difference.

Predicting a missing value

Assume a quadratic pattern: y = 3, 6, 11, 18, ? at unit spacing.

First differences	3, 5, 7; the next difference should be 9.
Second differences	5−3 = 2; 7−5 = 2. Assume this remains constant.
Prediction	Next y = 18 + 9 = 27. This depends on the quadratic assumption.

Forward versus backward selection

The difference table is the input to Newton interpolation.

Near the beginning	Use y_0, Δy_0, $\Delta^2 y_0$, … in Newton forward interpolation.
Near the end	Use y_n, ∇y_n, $\nabla^2 y_n$, … in Newton backward interpolation.
Identity	$\nabla^k y_i = \Delta^k y_{i-k}$; both tables contain the same differences with different alignment.

Algorithm and Python implementation

Generate successive forward difference columns.

Initialize

cols = [list(y)]

Repeat

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while len(cols[-1]) > 1:
    v = cols[-1]
    cols.append([b-a for a,b in zip(v,v[1:])])
```

Output

For y = [0,1,4,9,16], columns are [1,3,5,7], [2,2,2], [0,0], [0].
Cost: $O(n^2)$ time and storage.

Practice questions

Use the definitions and construct difference tables.

Q1	For y = [2, 5, 10, 17, 26], compute all forward difference columns.
Q2	For x = [0, 2, 4, 6], y = [0, 8, 64, 216], find the third difference.
Q3	For Q1, compute ∇y_4 and $\nabla^2 y_4$ using zero-based indices.

Solutions and recap

Next lecture: Newton Forward Difference Interpolation.

A1	Δ: [3,5,7,9]; Δ^2: [2,2,2]; Δ^3: [0,0]; Δ^4: [0].
A2	Δ: [8,56,152]; Δ^2: [48,96]; Δ^3: [48] ($3! \times 2^3$).
A3	$\nabla y_4 = 26-17 = 9$; $\nabla^2 y_4 = 9-7 = 2$.